Analitical approximation of $\langle\phi^2\rangle$ for a massive scalar field in static spherically symmetric spacetimes

Sergey V. Sushkov*
Department of Geometry, Kazan State Pedagogical University,
Mezhlauk 1 st., Kazan 420021, Russia

An analitical approximation of $\langle\phi^2\rangle$ for a massive scalar field in a zero temperature vacuum state in static spherically symmetric spacetimes is obtained. The calculations are based on the method for computing vacuum expectations values for scalar fields in general static spherically symmetric spacetimes derived by Anderson, Hiscock and Samuel [1,2]. The analitical approximation is used to compute $\langle\phi^2\rangle$ in Schwarzschild and wormhole spacetimes.

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I. INTRODUCTION

In the absence of a fully satisfactory theory of quantum gravity the semiclassical theory of gravity takes an important role in studies of effects of the back reaction of the quantized fields upon the spacetime geometry. A primary technical difficulty in semiclassical gravity is that expectation values have the strong functional dependence on the metric tensor $g_{\mu\nu}$ and are generally impossible to be calculated analitically. For this reason, much efforts have been concentrated on developing approximate methods.

In 1982, Page [3] developed a method for obtaining an approximate expression for $\langle T_{\mu\nu}\rangle$ for a conformal massless scalar field in static Einstein spaces. Later a slightly different approach was proposed by Brown and Ottewill [4] (see also Ref. [5]). In 1987, Frolov and Zel'nikov [6] constructed the approximate expression for $\langle T_{\mu\nu}\rangle$ for conformal massless fields in general static spacetimes; their construction was based on pure geometrical arguments and common properties of the stress-energy tensor. The effective action approach had been elaborated by a number of authors [7–15]. Recently, Anderson [1] and Anderson, Hiscock and Samuel [2] developed a method of deriving an approximation for $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ in a static spherically symmetric spacetime using the WKB approximation for the modes of the scalar field; the method is suitable for the fields with an arbitrary coupling ξ to the scalar curvature; also the fields can be either in a zero temperature vacuum state or a nonzero temperature thermal state.

The aim of this paper is to obtain the approximate expression for $\langle\phi^2\rangle$ using the Anderson-Hiskock-Samuel approach. Note that a practical applying of the method faces with considerable technical difficulty. It consists in necessity of computing the mode sums and then expanding them in powers of ϵ (where $\epsilon = i(t-t')$ characterizes the point splitting). In Refs. [1,2] such calculations was made approximately in the large ω limit. In this paper we present exact results on computing the mode sums.

Another difficulty has no computational nature. The problem is that a zeroth-order solution, which determines an iterative procedure of solving the mode equation, can be only defined up to terms of the second order. Of course, in principle, a final result of the iterative procedure should not depend on a particular choice of the zeroth-order solution. However, the form of an approximate solution of the second or higher order should depend on this choice. In this paper we solve the mode equation iteratively using the zeroth-order solution which is slightly different from that in Refs. [1,2] used.

The paper is organized as follows. In Sec. II, following the Anderson-Hiscock-Samuel approach, we develop the second-order WKB approximation for an unrenormalized expression for $\langle\phi^2\rangle$, where ϕ is a massive scalar field with arbitrary curvature coupling in a general static spherically symmetric spacetime. In Sec. III the resulting expression is renormalized using the method of covariant point splitting. In Secs. IV and V we apply the resulting analitical approximation for $\langle\phi^2\rangle$ to investigate the vacuum polarization of a massive scalar field in a Schwarzschild spacetime and in a wormhole spacetime, respectively. In Appendix A we derive the asymptotical expansions for the mode sums and integrals in the unrenormalized expression for ϕ. In Appendix B some technical details are discussed.

The units $\hbar = c = G = 1$ are used throughout the paper.

*e-mail: sushkov@kspu.kcn.ru

II. AN UNRENORMALIZED SECOND-ORDER WKB APPROXIMATION FOR $\langle\phi^2\rangle$

In this section we derive an unrenormalized expression for $\langle\phi^2\rangle$ for a massive scalar field in an arbitrary static spherically symmetric spacetime. Note that at this stage our consideration will mainly follow the papers by Anderson [1] and Anderson, Hiscock, and Samuel [2] in which one may find some additional details.

As in Refs. [1,2], a Euclidean space approach is used. The metric for a general static spherically symmetric spacetime when continued analitically into Euclidean space is

$$ds^2 = f(r)d\tau^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{1}$$

Here $\tau = it$ is the Euclidean time, and f and h are arbitrary functions of r which, if the space is asymptotically flat, become constant in the limit $r \to \infty$.

Consider a quantized scalar field ϕ with mass m and coupling ξ to the scalar curvature R. We assume that the field is in a vacuum state defined with respect to the Killing vector which always exists in a static spacetime. An unrenormalized expression for $\langle\phi^2\rangle$ can be computed from the Euclidean Green's function $G_E(x,\tilde{x})$. This expression given in Refs. [1,2] is

$$\begin{aligned}\langle\phi^2(x)\rangle_{\text{unren}} &= G_E(x,\tau;x,\tilde{\tau}) \\ &= \frac{1}{4\pi^2}\int_0^\infty d\omega \cos[\omega(\tau-\tilde{\tau})] \sum_{l=0}^{\infty}\left[(2l+1)p_{\omega l}q_{\omega l} - \frac{1}{rf^{1/2}}\right],\end{aligned} \tag{2}$$

where the modes $p_{\omega l}$ and $q_{\omega l}$ obey the equation

$$\frac{1}{h}\frac{d^2S}{dr^2} + \left[\frac{2}{rh} + \frac{1}{2fh}\frac{df}{dr} - \frac{1}{2h^2}\frac{dh}{dr}\right]\frac{dS}{dr} - \left[\frac{\omega^2}{f} + \frac{l(l+1)}{r^2} + \xi R\right]S = 0. \tag{3}$$

They also satisfy the Wronskian condition

$$C_{\omega l}\left[p_{\omega l}\frac{dq_{\omega l}}{dr} - q_{\omega l}\frac{dp_{\omega l}}{dr}\right] = -\frac{1}{r^2}\left(\frac{h}{f}\right)^{1/2}, \tag{4}$$

where $C_{\omega l}$ is a normalization constant.

Let us stress that points in Eq. (2) are splitted. Namely, points are separated in time so that $\epsilon \equiv (\tau - \tilde{\tau})$, $\tilde{r} = r$, $\tilde{\theta} = \theta$, $\tilde{\varphi} = \varphi$. As was first pointed out by Candelas and Howard [16] for the case of Schwarzschild spacetime, the Euclidean Green's function have superficial divergences with this separation of points. As discussed in Refs. [1,2], these cannot be real divergences because the Green's function must be finite when the points are separated; to remove the divergences one has to subtract some additional counterterms. The terms $(rf^{1/2})^{-1}$ in brackets in Eq. (2) are such counterterms (for details, see discussion in Refs. [1,2]).

There is a WKB representation for the modes which is very useful in calculations of $\langle\phi^2\rangle$. The WKB representation is obtained by the change of variables

$$\begin{aligned} p_{\omega l} &= \frac{1}{(2r^2W)^{1/2}}\exp\left[\int^r W\left(\frac{h}{f}\right)^{1/2} dr\right], \\ q_{\omega l} &= \frac{1}{(2r^2W)^{1/2}}\exp\left\{-\left[\int^r W\left(\frac{h}{f}\right)^{1/2} dr\right]\right\}, \end{aligned} \tag{5}$$

where W is a new function of r. Substitution of Eq. (5) into Eq. (4) shows that the Wronskian condition is obeyed if $C_{\omega l} = 1$, and substitution of Eq. (5) into Eq. (3) gives the following equation for W:

$$W^2 = \omega^2 + m^2 f + l(l+1)\frac{f}{r^2} + \frac{1}{2r}\left(\frac{f}{h}\right)' + \xi R f + \frac{1}{2}\frac{f}{h}\frac{W''}{W} + \frac{1}{4}\left(\frac{f}{h}\right)'\frac{W'}{W} - \frac{3}{4}\frac{f}{h}\frac{W'^2}{W^2}, \tag{6}$$

where the scalar curvature R is

$$R = -\frac{f''}{fh} + \frac{f'^2}{2f^2h} + \frac{f'h'}{2fh^2} - \frac{2f'}{rfh} + \frac{2h'}{rh^2} - \frac{2}{r^2h} + \frac{2}{r^2},$$

and the prime denotes the derivative with respect to r. The usual way to work out Eq. (6) is to solve it iteratively. It is obvious that a choice of zeroth-order iteration determines the further iterative procedure. In Refs. [1,2] it was suggested to take

$$\overline{W}^{(0)} = \overline{\Omega}(r) = \left[\omega^2 + m^2 f + (l+1/2)^2 \frac{f}{r^2}\right]^{1/2}$$

as the zeroth-order solution. This choice seems to be convenient because it simplifies some further calculations. However, this is not the best choice. Really, consider the asymptotically flat region of spacetime where $f, h \to 1$ and $f', f'', h', h'' \to 0$. Assume also that $W', W'' \to 0$ there. In this case the equation (6) reduces to

$$W^2 = \Omega^2(r)$$

with

$$\Omega(r) = \left[\omega^2 + m^2 f + l(l+1)\frac{f}{r^2}\right]^{1/2}. \tag{7}$$

Hence, it is seen that a natural choice for the zeroth-order solution is $W^{(0)} = \Omega$. Note that $W^{(0)} = \overline{W}^{(0)} - f(4r^2)^{-1}$.

So, we take $W^{(0)} = \Omega$ as the zeroth-order solution. Then, the second-order solution of Eq. (6) is

$$W = \Omega + W^{(2)} = \Omega + \frac{V_1 + V_2}{2\Omega} + \frac{1}{4}\frac{f}{h}\frac{\Omega''}{\Omega^2} + \frac{1}{8}\left(\frac{f}{h}\right)' \frac{\Omega'}{\Omega^2} - \frac{3}{8}\frac{f}{h}\frac{\Omega'^2}{\Omega^3}, \tag{8}$$

with

$$V_1(r) = \frac{1}{2r}\left(\frac{f}{h}\right)', \quad V_2(r) = \xi R f.$$

Discuss the question about applicability of the second-order solution. It is correct provided $\Omega \gg |W^{(2)}|$. This condition must be fulfilled for each mode. In particular, for the lowest null mode with $\omega = 0$ and $l = 0$ we obtain

$$m^2 \gg \left|\frac{1}{2}\xi R + \frac{1}{4}\frac{f'}{fhr} - \frac{1}{4}\frac{h'}{h^2 r} + \frac{1}{8}\frac{f''}{fh} - \frac{1}{16}\frac{f'h'}{fh^2} - \frac{3}{32}\frac{f'^2}{f^2 h}\right|. \tag{9}$$

In the other words, the last condition means that the Compton length $r_c = m^{-1}$ is much smaller than the characteristic length scale of the spacetime geometry. Restrictions, which the inequality (9) imposes for the metric functions f and h, ensure correctness of the WKB approximation. Hereinafter we shall suppose that f and h answer these requirements.

To obtain the second-order WKB approximation for $\langle\phi^2\rangle$ we substitute Eq. (8) into (5), and then into (2). Neglecting terms of the fourth order and higher we can finally find

$$\begin{aligned}\langle\phi^2\rangle_{\text{unren}} = \frac{1}{4\pi^2}\int_0^\infty d\omega \cos[\omega(\tau-\tau')] \sum_{l=0}^{\infty} \Bigg\{ \left(\frac{l+1/2}{r^2\Omega} - \frac{1}{r\sqrt{f}}\right) - \frac{l+1/2}{r^2\Omega^3}\frac{V_1+V_2}{2} \\ -\frac{1}{8}\frac{l+1/2}{r^2\Omega^5}\left[\frac{f}{h}U'' + \frac{1}{2}\left(\frac{f}{h}\right)' U'\right] + \frac{5}{32}\frac{l+1/2}{r^2\Omega^7}\frac{f}{h}U'^2\Bigg\},\end{aligned} \tag{10}$$

with

$$U(r) = m^2 f + l(l+1)\frac{f}{r^2}.$$

III. A RENORMALIZED EXPRESSION FOR $\langle\phi^2\rangle$

In this section we derive a renormalized expression for $\langle\phi^2\rangle$ in the framework of the second-order WKB approximation.

As is usual for the method of point splitting, in order to renormalize $\langle\phi^2\rangle_{\text{unren}}$ one should subtruct renormalization counterterms from $\langle\phi^2\rangle_{\text{unren}}$ and take the limit $\epsilon \to 0$ ($\tau' \to \tau$). Schematically,

$$\langle\phi^2\rangle_{\rm ren} = \lim_{\epsilon\to 0}\left(\langle\phi^2\rangle_{\rm unren} - \langle\phi^2\rangle_{\rm DS}\right). \tag{11}$$

The renormalization counterterms for a scalar field has been obtained by Christensen [17] which used the DeWitt-Schwinger expansion for the Feynman Green's function. In the case of a massive scalar field they are given by

$$\begin{aligned}\langle\phi^2\rangle_{\rm DS} &= G_{\rm DS}(x,x')\\ &= \frac{1}{8\pi^2\sigma} + \frac{1}{8\pi^2}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\left[C + \frac{1}{2}\ln\left(\frac{m^2|\sigma|}{2}\right)\right] - \frac{m^2}{16\pi^2} + \frac{1}{96\pi^2}R_{\alpha\beta}\frac{\sigma^\alpha\sigma^\beta}{\sigma}.\end{aligned} \tag{12}$$

Here σ is equal to one half the square of the distance between the points x and x' along the shortest geodesic connecting them. C is Euler's constant, $R_{\alpha\beta}$ is the Ricci tensor and $\sigma^\alpha \equiv \sigma^{;\alpha}$.

To perform the procedure of renormalization in practice, one should expand the expressions for $\langle\phi^2\rangle_{\rm unren}$ and $\langle\phi^2\rangle_{\rm DS}$ in powers of ϵ. The expansion of $\langle\phi^2\rangle_{\rm DS}$ can be easily found by using the expansion of σ and its derivatives in powers of ϵ given in Refs. [1,2]:

$$\begin{aligned}&\sigma = \frac{1}{2}f\epsilon^2 + {\rm O}(\epsilon^4), \quad \sigma^\tau = -\epsilon + \frac{f'^2}{24fh}\epsilon^3 + {\rm O}(\epsilon^5),\\ &\sigma^r = \frac{f'}{4h}\epsilon^2 + {\rm O}(\epsilon^4), \quad \sigma^\theta = \sigma^\phi = 0.\end{aligned} \tag{13}$$

In general, we have $\langle\phi^2\rangle_{\rm DS} = a\epsilon^{-2} + b\ln\epsilon + c + O(\epsilon^2)$ where first two terms describe ultraviolet divergences. It is well-known that the second-order WKB approximation for $\langle\phi^2\rangle_{\rm unren}$ contains the same divergences, and so the subtraction of $\langle\phi^2\rangle_{\rm DS}$ annihilates terms of $\langle\phi^2\rangle_{\rm unren}$ that is infinite in the limit $\epsilon\to 0$.

To obtain the expansion of $\langle\phi^2\rangle_{\rm unren}$ in powers of ϵ we rewrite Eq. (10) in the form being more convenient for further analysis:

$$\begin{aligned}4\pi^2\langle\phi^2\rangle_{\rm unren} = &\frac{1}{r^2}S_0(\epsilon,\mu) - S_1(\epsilon,\mu)\frac{V_1+V_2}{2f} - S_2(\epsilon,\mu)\frac{r^2}{8f^2}\left[\frac{f}{h}\left(\frac{f}{r^2}\right)'' + \frac{1}{2}\left(\frac{f}{h}\right)'\left(\frac{f}{r^2}\right)'\right]\\ &+S_3(\epsilon,\mu)\frac{5r^4}{32f^2h}\left(\frac{f}{r^2}\right)'^2 - N_2^1(\epsilon,\mu)\frac{r^2}{8f^2}\left[\frac{f}{h}\left(\frac{f}{r^2}\mu^2\right)'' + \frac{1}{2}\left(\frac{f}{h}\right)'\left(\frac{f}{r^2}\mu^2\right)'\right]\\ &+N_3^1(\epsilon,\mu)\frac{5r^4}{32f^2h}\left(\frac{f}{r^2}\mu^2\right)'^2 + N_3^3(\epsilon,\mu)\frac{5r^4}{16f^2h}\left(\frac{f}{r^2}\right)'\left(\frac{f}{r^2}\mu^2\right)'\end{aligned} \tag{14}$$

with

$$S_0(\epsilon,\mu) = \int_0^\infty du\cos\left(u\frac{\epsilon\sqrt{f}}{r}\right)\sum_{l=0}^\infty\left[\frac{l+1/2}{\sqrt{u^2+\mu^2+(l+1/2)^2}} - 1\right], \tag{15.1}$$

$$\begin{aligned}&S_n(\epsilon,\mu) = \int_0^\infty du\cos\left(u\frac{\epsilon\sqrt{f}}{r}\right)\sum_{l=0}^\infty\frac{(l+1/2)^{2n-1}}{[u^2+\mu^2+(l+1/2)^2]^{n+1/2}},\\ &n = 1,2,3,\ldots,\end{aligned} \tag{15.2}$$

$$\begin{aligned}&N_n^m(\epsilon,\mu) = \int_0^\infty du\cos\left(u\frac{\epsilon\sqrt{f}}{r}\right)\sum_{l=0}^\infty\frac{(l+1/2)^m}{[u^2+\mu^2+(l+1/2)^2]^{n+1/2}},\\ &n = 1,2,3,\ldots,\\ &m = 1,3,\ldots,2n-3\end{aligned} \tag{15.3}$$

where we denote

$$\mu^2 \equiv m^2r^2 - \frac{1}{4}. \tag{16}$$

Note that $\mu^2 > 0$ if $r > (2m)^{-1}$.

Now one must calculate the asymptotical expansion for the functions $S_0(\epsilon,\mu)$, $S_n(\epsilon,\mu)$ and $N_n^m(\epsilon,\mu)$ in the limit $\epsilon\to 0$. In Appendix A details of such calculations are given. Here we present some final formulas:

$$S_0(\epsilon,\mu)=\frac{r^2}{f\epsilon}+\frac{1}{2}\left(\mu^2-\frac{1}{12}\right)\left(\ln\frac{\epsilon\mu}{2}+C\right)-\frac{\mu^2}{2}-\mu^2 S_0(2\pi\mu)+\mathrm{O}(\epsilon^2\ln\epsilon), \tag{17}$$

$$S_n(\epsilon,\mu)=-\frac{2^{n-1}(n-1)!}{(2n-1)!!}\left(\ln\frac{\epsilon\mu}{2}+C\right)+\frac{S_n(2\pi\mu)}{(2n-1)!!}+\mathrm{O}(\epsilon^2\ln\epsilon),$$

where $(2n-1)!!\equiv 1\cdot 3\cdot 5\cdots(2n-1)$, $n=1,2,3,\dots$ and $S_0(2\pi\mu)$ and $S_n(2\pi\mu)$ are determined by Eqs. (A.14).

Note that the functions $N_n^m(\epsilon,\mu)$ are regular in the limit $\epsilon\to 0$. Hence one may directly put $\epsilon=0$, so that

$$N_m^m(0,\mu)=\int_0^\infty du\sum_{l=0}^{\infty}\frac{(l+\frac{1}{2})^m}{[u^2+\mu^2+(l+\frac{1}{2})^2]^{n+1/2}}. \tag{18}$$

Changing the order of summation and integration in Eq. (18) and integrating over u gives, in particular,

$$N_2^1(0,\mu)\equiv\frac{2}{3}N_2^1(\mu)=\frac{2}{3}\sum_{l=0}^{\infty}\frac{l+\frac{1}{2}}{[\mu^2+(l+\frac{1}{2})^2]^2}, \tag{19.1}$$

$$N_3^i(0,\mu)\equiv\frac{8}{15}N_3^i(\mu)=\frac{8}{15}\sum_{l=0}^{\infty}\frac{(l+\frac{1}{2})^i}{[\mu^2+(l+\frac{1}{2})^2]^3}, \tag{19.2}$$

where $i=1,3$.

Note also that the functions $S_n(2\pi\mu)$ and $N_n^m(\mu)$ have a simple asymptotical form at large values of μ, or, correspondingly, at large values of mr (see Appendix B for details):

$$\begin{aligned}
&S_0(2\pi\mu)\approx-\frac{7}{1920(mr)^4},\quad S_1(2\pi\mu)\approx\frac{1}{24(mr)^2},\\
&S_2(2\pi\mu)\approx-\frac{0.0146}{(mr)^4},\quad S_3(2\pi\mu)\approx\frac{0.03}{(mr)^6},\\
&N_2^1(\mu)\approx\frac{1}{2(mr)^2},\quad N_3^1(\mu)\approx\frac{1}{4(mr)^4},\\
&N_3^3(\mu)\approx\frac{1}{4(mr)^2}.
\end{aligned} \tag{20}$$

Substituting Eqs. (17,19.1) into Eq. (14) we may find the asymptotical expansion for $\langle\phi^2\rangle_{\rm unren}$ in the limit $\epsilon\to 0$. Finally, carring out the procedure of renormalization (11), i.e. subtracting $\langle\phi^2\rangle_{\rm DS}$, we obtain the renormalized expression for $\langle\phi^2\rangle$ in the framework of the second-order WKB approximation:

$$\begin{aligned}
4\pi^2\langle\phi^2\rangle_{\rm ren}=&\frac{1}{4}\left[m^2+(\xi-\frac{1}{6})R\right]\ln\left(1-\frac{1}{4m^2r^2}\right)+\frac{1}{16r^2}+\frac{f''}{24fh}-\frac{f'^2}{24f^2}-\frac{f'h'}{48fh^2}+\frac{f'}{12fhr}\\
&-m^2S_0(2\pi\mu)\left(1-\frac{1}{4m^2r^2}\right)-S_1(2\pi\mu)\frac{V_1+V_2}{2f}-S_2(2\pi\mu)\frac{r^2}{24f^2}\left[\frac{f}{h}\left(\frac{f}{r^2}\right)''+\frac{1}{2}\left(\frac{f}{h}\right)'\left(\frac{f}{r^2}\right)'\right]\\
&+S_3(2\pi\mu)\frac{r^4}{96f^2h}\left(\frac{f}{r^2}\right)'^2-N_2^1(\mu)\frac{r^2}{8f^2}\left[\frac{f}{h}\left(\frac{f}{r^2}\mu^2\right)''+\frac{1}{2}\left(\frac{f}{h}\right)'\left(\frac{f}{r^2}\mu^2\right)'\right]\\
&+N_3^1(\mu)\frac{5r^4}{32f^2h}\left(\frac{f}{r^2}\mu^2\right)'^2+N_3^3(\mu)\frac{5r^4}{16f^2h}\left(\frac{f}{r^2}\right)'\left(\frac{f}{r^2}\mu^2\right)'.
\end{aligned} \tag{21}$$

IV. $\langle\phi^2\rangle$ IN A SCHWARZSCHILD SPACETIME

In this section we apply the analitical approximation for $\langle\phi^2\rangle$, obtained above, to investigate the vacuum polarization of a massive scalar field with arbitrary curvature coupling in a Schwarzschild spacetime.

For a Schwarzschild spacetime the metric functions f and h are

$$f = h^{-1} = 1 - \frac{2M}{r}, \tag{22}$$

where M is the mass of the black hole. The event horizon is at

$$r_{\rm g} = 2M. \tag{23}$$

The scalar curvature R is identically zero for a Schwarzschild spacetime.

We shall compute $\langle\phi^2\rangle$ in the region exterior to the event horizon where the spacetime is static, i.e. $r > r_{\rm g}$. The condition (9), ensuring applicability of the WKB approximation in this region, can be rewritten as

$$\left(\frac{r_{\rm g}}{r_{\rm c}}\right)^2 \gg \left|\frac{1}{4\rho^3} - \frac{1}{32\rho^4(1-\rho^{-1})}\right|, \tag{24}$$

where $r_{\rm c} = m^{-1}$ is Compton radius corresponding to a scalar field with the mass m, and $\rho = r/r_{\rm g}$ is the dimensionless coordinate, $\rho > 1$. For example, let us take $r_{\rm g} = 2M_\odot = 1.48\cdot 10^5$cm and $r_{\rm c} = m_{\rm e}^{-1} = 3.86\cdot 10^{-11}$cm where $M_\odot$ is the Sun mass and $m_{\rm e}$ is the electron mass. In this case the inequality (24) is fulfilled for $\rho - 1 > 10^{-30}$.

We shall assume that $r_{\rm g}/r_{\rm c} \gg 1$, and so $mr = (r_{\rm g}/r_{\rm c})\rho \gg 1$ because of $\rho > 1$. Therefore, the functions S_n and N_n^m can be taken in their asymptotical form (20) and the expression (21) for $\langle\phi^2\rangle$ can be written down as follows:

$$\begin{aligned}16\pi^2 M^2\langle\phi^2\rangle_{\rm ren} = {}& \frac{1}{4}\Lambda^2\ln\left(1-\frac{1}{4\Lambda^2\rho^2}\right) + \frac{1}{16\rho^2} + \frac{7}{1920}\frac{1}{\Lambda^2\rho^4} - \frac{1}{48}\frac{1}{\Lambda^2\rho^5} - \frac{7}{7680}\frac{1}{\Lambda^4\rho^6} \\ & - \frac{a_1}{\Lambda^4\rho^6(1-\rho^{-1})^2}\left(1 - \frac{13}{3\rho} + \frac{35}{6\rho^2} - \frac{5}{2\rho^3}\right) + \frac{a_2}{\Lambda^6\rho^8(1-\rho^{-1})}\left(1 - \frac{3}{\rho} + \frac{9}{4\rho^2}\right),\end{aligned} \tag{25}$$

where $\Lambda \equiv 2Mm = r_{\rm g}/r_{\rm c}$, $a_1 = 3,646\cdot 10^{-3}$ and $a_2 = 1.281\cdot 10^{-3}$. Note also that at large values of ρ, i.e. far from the event horizon, the last expression has a simple asymptotical form. Taking into account that $\ln(1-\frac{1}{x}) \approx -(\frac{1}{x} + \frac{1}{2x^2})$ for large values of x we obtain

$$16\pi^2 M^2\langle\phi^2\rangle_{\rm ren} \approx -\frac{1}{240\Lambda^2\rho^4}. \tag{26}$$

Some results of numerical calculations using the analitical approximation (25) are shown in Fig. 1.

V. $\langle\phi^2\rangle$ IN A WORMHOLE SPACETIME

In this section we apply the analitical approximation (21) to calculate $\langle\phi^2\rangle$ for a massive scalar field in a wormhole spacetime.

Here we shall regard a wormhole as a time independent, nonrotating, and spherically symmetric bridge connecting two asymptotically flat regions. The metric of wormhole spacetime (continued analitically into Euclidean space) can be taken in the form, suggested by Morris and Thorne [18]:

$$ds^2 = f(l)d\tau^2 + dl^2 + r^2(l)(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{27}$$

where l is the proper radial distance, $l \in (-\infty, +\infty)$. We assume that the redshift function $f(l)$ is everywhere finite (no event horizons); the shape function $r(l)$ has the global minimum at $l = 0$, so that $r_0 = \min\{r(l)\} = r(0)$ is the radius of the wormhole throat. In order for the spacetime geometry to tend to an appropriate asymptotically flat limit at $l \to \pm\infty$ we impose

$$\lim_{l\to\pm\infty}\{r(l)/|l|\} = 1, \quad \text{and} \quad \lim_{l\to\pm\infty} f(l) = f_\pm. \tag{28}$$

For simplicity we also assume symmetry under interchange of the two asymptotically flat regions, $l \leftrightarrow -l$, that is, $r(l) = r(-l)$ and $f(l) = f(-l)$.

Note that two metrics, (1) and (27), can be derived from each other by substitution

$$\frac{dr}{dl} = \pm\frac{1}{\sqrt{h}}. \tag{29}$$

Now

$$\frac{1}{h} = r'^2, \quad \frac{1}{h}\frac{dh}{dr} = r'', \tag{30}$$

where the prime denotes the derivative with respect to l.

Let ϕ be a massive scalar field with minimal curvature coupling ($\xi = 0$) in the wormhole spacetime. Consider the simplest example, given in Ref. [18], when the metric functions $f(l)$ and $r(l)$ are

$$f(l) \equiv 1, \quad r(l) = \sqrt{l^2 + r_0^2}. \tag{31}$$

The condition (9), ensuring applicability of the WKB approximation in the wormhole spacetime, reduces in this case to

$$m^2 \gg \frac{1}{2}\frac{r''}{r} = \frac{1}{2}\frac{r_0^2}{(l^2 + r_0^2)}. \tag{32}$$

It is seen that this inequality is satisfied for all values of l provided $r_0^2 \gg (2m)^{-1} = r_{\rm c}/2$. This means that the throat's radius has to be much greater than Compton radius. If so, then $mr \gg 1$, and the functions S_n and N_n^m can be taken in their asymptotical form (20). Now the analitical approximation (21) for $\langle\phi^2\rangle$ can be written as

$$\begin{aligned} 16\pi^2 r_{\rm c}^2 \langle\phi^2\rangle_{\rm ren} = & \left[1 + \frac{1}{3}\frac{\rho_0^2}{(\rho^2+\rho_0^2)^2}\right] \ln\left(1 - \frac{1}{4}\frac{1}{\rho^2+\rho_0^2}\right) + \frac{1}{4}\frac{1}{\rho^2+\rho_0^2} - \frac{b_1}{(\rho^2+\rho_0^2)^2} + \frac{b_2}{(\rho^2+\rho_0^2)^3} \\ & + \frac{\rho^2}{(\rho^2+\rho_0^2)^3}\left[b_3 + \frac{b_4}{\rho^2+\rho_0^2} + \frac{b_5}{(\rho^2+\rho_0^2)^2}\right], \end{aligned} \tag{33}$$

where $\rho = ml$ is the dimensionless proper radial distance, $\rho_0 = mr_0$ is the dimensionless throat's radius, and the numbers b_i are $b_1 = \frac{31}{160}$, $b_2 = 8.51 \cdot 10^{-3}$, $b_3 = \frac{13}{48}$, $b_4 = 5.851 \cdot 10^{-2}$, $b_5 = 5.126 \cdot 10^{-3}$. Note also that far from the wormhole's throat, $\rho \to \infty$, the analitical approximation (33) has the simple asymptotical form:

$$16\pi^2 r_{\rm c}^2 \langle\phi^2\rangle_{\rm ren} \approx \frac{11}{240\rho^4}, \tag{34}$$

which does not depend on the throat's radius.

In Fig. 2 we present some results of numerical calculations obtained by using the analitical approximation (33).

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APPENDIX A: ASYMPTOTICAL EXPANSION FOR $S_N(\epsilon, \mu)$ IN THE LIMIT $\epsilon \to 0$

In this appendix we derive the asymptotical expansion for the functions $S_0(\epsilon, \mu)$ and $S_n(\epsilon, \mu)$ in the limit $\epsilon \to 0$. Consider the following sums over l:

$$s_0(y) = \sum_{l=0}^{\infty}\left[\frac{l+1/2}{\sqrt{y^2+(l+1/2)^2}} - 1\right], \tag{A.1}$$

$$s_n(y) = \sum_{l=0}^{\infty}\frac{(l+1/2)^{2n-1}}{[y^2+(l+1/2)^2]^{n+1/2}}, \tag{A.2}$$

where $n = 1, 2, 3, \ldots$ and $y > 0$. To transform them we use the Abel-Plana method of summarizing of convergent series [19]. The essence of this method consists of following: the series is presented as a contour integral

$$2i \sum_n f(n + \frac{1}{2}) = 2\pi i \sum_n \operatorname*{res}_{z=n+\frac{1}{2}} \left[f(z) \cot \pi (z - \frac{1}{2}) \right] = \int_{\mathcal{C}} f(z) \cot \pi (z - \frac{1}{2}) dz\,, \tag{A.3}$$

where $\mathcal{C}$ is a contour which surrounds a region on the complex plane containing only the poles of $\cot \pi (z - \frac{1}{2})$; the poles of $f(z)$ (if they exist) have to lie outside of the contour $\mathcal{C}$. Deforming the contour by a special way and currying out not difficult algebraic transformations gives the following relation:

$$\sum_{l=0}^{\infty} f(l + \frac{1}{2}) = \lim_{\sigma \to 0} \left\{ \int_{\sigma}^{\infty} f(x) dx \; + i \int_{i\sigma}^{i\sigma + \infty} \frac{f(-it) dt}{e^{2\pi t} + 1} - i \int_{-i\sigma}^{-i\sigma + \infty} \frac{f(it) dt}{e^{2\pi t} + 1} \right\}. \tag{A.4}$$

Using this formula we may transform Eqs. (A.1) and (A.2) as follows:

$$s_0(y) = \int_0^{\infty} \left[\frac{x}{\sqrt{y^2 + x^2}} - 1 \right] dx + 2 \int_0^{y} \frac{t}{\sqrt{y^2 - t^2}} \frac{dt}{e^{2\pi t} + 1}, \tag{A.5}$$

$$s_n(y) = \int_0^{\infty} \frac{x^{2n-1} dx}{(y^2 + x^2)^{n+1/2}} - 2(-1)^n \,\rlap{\,-}\!\!\int_0^{y} \frac{t^{2n-1}}{(y^2 - t^2)^{n+1/2}} \frac{dt}{e^{2\pi t} + 1}, \tag{A.6}$$

where

$$\rlap{\,-}\!\!\int_0^{y} f(t) dt = \lim_{\delta \to 0} \left[\int_0^{y - \delta} f(t) dt - \begin{pmatrix} \text{terms which diverge} \\ \text{in the limit } \delta \to 0 \end{pmatrix} \right].$$

Calculating the first integral in Eq. (A.5) and carring out an integration by parts in Eq. (A.6) yields

$$s_0(y) = -y + 2 \int_0^{y} \frac{t}{\sqrt{y^2 - t^2}} \frac{dt}{e^{2\pi t} + 1}, \tag{A.7}$$

$$s_n(y) = -\frac{2}{(2n-1)!!} \int_0^{y} \frac{dt}{\sqrt{y^2 - t^2}} \frac{d}{dt} \left(\frac{1}{t} \frac{d}{dt} \right)^{n-1} \left(\frac{t^{2(n-1)}}{e^{2\pi t} + 1} \right). \tag{A.8}$$

The expressions for the functions $S_n(\epsilon, \mu)$, given by Eqs. (15.1) and (15.2), may be rewritten as

$$S_n(\epsilon, \mu) = \int_0^{\infty} du \cos(u\varepsilon) s_n(\sqrt{u^2 + \mu^2}),$$

where $\varepsilon = \epsilon r^{-1} f^{1/2}$ and $n = 0, 1, 2, \ldots$.

In order to derive the asymptotical expansion for $S_0(\epsilon, \mu)$ and $S_n(\epsilon, \mu)$ in the limit $\epsilon \to 0$ we use the following asymptotical relations:

$$I_1(\varepsilon) \equiv \int_0^{\infty} \sqrt{u^2 + \mu^2} \cos(u\varepsilon) du = -\frac{1}{\varepsilon^2} - \frac{\mu^2}{2} \left(\ln \frac{\varepsilon \mu}{2} + C \right) + \frac{\mu^2}{4} + \mathrm{O}(\varepsilon^2 \ln \varepsilon), \tag{A.9}$$

$$\begin{aligned} I_2(\varepsilon) &\equiv \int_0^{\infty} du \, \cos(u\varepsilon) \int_0^{\sqrt{u^2+\mu^2}} \frac{f(t) dt}{\sqrt{u^2 + \mu^2 - t^2}} \\ &= -\left(\ln \frac{\varepsilon}{2} + C \right) \int_0^{\infty} f(t)\, dt - \frac{1}{2} \int_0^{\infty} f(t) \ln |\mu^2 - t^2|\, dt + \mathrm{O}(\varepsilon^2 \ln \varepsilon). \end{aligned} \tag{A.10}$$

To prove the first formula (A.9) we make the following transformations:

$$
\begin{aligned}
I_1(\varepsilon) &= \int_0^\infty \frac{(u^2+\mu^2)\cos(u\varepsilon)}{\sqrt{u^2+\mu^2}}\,du \\
&= \left(-\frac{d^2}{d\varepsilon^2}+\mu^2\right)\int_0^\infty \frac{\cos(u\varepsilon)}{\sqrt{u^2+\mu^2}}\,du \\
&= \left(-\frac{d^2}{d\varepsilon^2}+\mu^2\right)K_0(\varepsilon\mu) \\
&= -\frac{\mu}{\varepsilon}K_1(\varepsilon\mu).
\end{aligned}
$$

Here $K_n(y)$ are the Bessel functions of the second kind. Taking into account the asymptotic of $K_1(y)$ in the limit $y \to 0$:

$$
K_1(y) = \frac{1}{y} + \frac{y}{2}\left(\ln\frac{y}{2} + C - \frac{1}{2}\right) + \mathrm{O}(y^2 \ln y),
$$

we obtain the relation (A.9).

To prove the second formula (A.10) we change the order of integration over u and t in I_2 and compute the integral over u, so that

$$
\begin{aligned}
I_2(\varepsilon) &= \int_0^\mu f(t)\,dt \int_0^\infty \frac{\cos(u\varepsilon)\,du}{\sqrt{u^2+\mu^2-t^2}} + \int_\mu^\infty f(t)\,dt \int_{\sqrt{t^2-\mu^2}}^\infty \frac{\cos(u\varepsilon)\,du}{\sqrt{u^2+\mu^2-t^2}} \\
&= \int_0^\mu f(t)K_0\left(\varepsilon\sqrt{\mu^2-t^2}\right)\,dt - \frac{\pi}{2}\int_\mu^\infty f(t)Y_0\left(\varepsilon\sqrt{t^2-\mu^2}\right)\,dt.
\end{aligned} \tag{A.11}
$$

Here $K_0(y)$ and $Y_0(y)$ are the Bessel functions of the second kind. Taking into account the asymptotics of K_0 and Y_0 in the limit $y \to 0$:

$$
\begin{aligned}
K_0(y) &= -(\ln\frac{y}{2} + C) + \mathrm{O}(y^2 \ln y), \\
Y_0(y) &= \frac{2}{\pi}(\ln\frac{y}{2} + C) + \mathrm{O}(y^2 \ln y),
\end{aligned}
$$

we can obtain the relation (A.10).

Finally, substituting Eqs. (A.7) and (A.8) into Eq. (A.9) and using formulas (A.9) and (A.10) gives

$$
S_0(\epsilon,\mu) = \frac{r^2}{f\epsilon} + \frac{1}{2}\left(\mu^2 - \frac{1}{12}\right)\left(\ln\frac{\epsilon\mu}{2} + C\right) - \frac{\mu^2}{2} - \mu^2 S_0(2\pi\mu) + \mathrm{O}(\epsilon^2\ln\epsilon), \tag{A.12}
$$

$$
S_n(\epsilon,\mu) = -\frac{2^{n-1}(n-1)!}{(2n-1)!!}\left(\ln\frac{\epsilon\mu}{2} + C\right) + \frac{S_n(2\pi\mu)}{(2n-1)!!} + \mathrm{O}(\epsilon^2\ln\epsilon), \tag{A.13}
$$

where $(2n-1)!! \equiv 1\cdot 3\cdot 5\cdots(2n-1)$, $n = 1, 2, 3, \ldots$ and $S_0(2\pi\mu)$ and $S_n(2\pi\mu)$ are

$$
\begin{aligned}
S_0(2\pi\mu) &= \int_0^\infty \frac{x\ln|1-x^2|}{e^{2\pi\mu x}+1}dx, \\
S_n(2\pi\mu) &= \int_0^\infty dx\,\ln|1-x^2|\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right)^{n-1}\frac{x^{2(n-1)}}{e^{2\pi\mu x}+1}.
\end{aligned} \tag{A.14}
$$

APPENDIX B: ASYMPTOTICS FOR $S_N(2\pi\mu)$ AND $N_N^M(\mu)$

In this appendix we obtain asymptotics at large values of argument for the functions $S_n(2\pi\mu)$ and $N_n^m(\mu)$. Consider the functions $S_n(2\pi\mu)$ defined by Eqs. (A.14). Making the substitution $y = 2\pi\mu x$ in Eqs. (A.14) we find

$$
\begin{aligned}
S_0(2\pi\mu) &= \lambda^2 \int_0^\infty \frac{y \ln|1-\lambda^2 y^2|}{e^y+1}\, dy, \\
S_n(2\pi\mu) &= \int_0^\infty dy \ln|1-\lambda^2 y^2| \frac{d}{dy}\left(\frac{1}{y}\frac{d}{dy}\right)^{n-1} \frac{y^{2(n-1)}}{e^y+1},
\end{aligned} \tag{B.1}
$$

where $\lambda = (2\pi\mu)^{-1}$ and $\mu = (m^2r^2 + \frac{1}{4})^{1/2}$. Note that the integrands in Eqs. (B.1) contain exponential functions and are exponentially decreasing at large values of y. Hence the main contribution into the integrals is provided with values of integrands in the region $0 < y < 1$. We are interesting in the case $\mu \gg 1$. In this case $\lambda \ll 1$ and $\lambda y \ll 1$ if $0 < y < 1$. Now we may use the asymptotical formula $\ln(1-\lambda^2y^2) = -\lambda^2y^2 - \frac{1}{2}\lambda^4y^4 - \frac{1}{3}\lambda^6y^6 + \mathrm{O}(\lambda^8y^8)$. Substituting this into Eqs. (B.1) and integrating gives

$$
\begin{aligned}
S_0(2\pi\mu) &= -\frac{7}{1920(mr)^4} + \mathrm{O}((mr)^{-6}), \qquad S_1(2\pi\mu) = \frac{1}{24(mr)^2} + \mathrm{O}((mr)^{-4}), \\
S_2(2\pi\mu) &= -\frac{0.0146}{(mr)^4} + \mathrm{O}((mr)^{-6}), \qquad S_3(2\pi\mu) = \frac{0.03}{(mr)^6} + \mathrm{O}((mr)^{-8}).
\end{aligned} \tag{B.2}
$$

Consider now the functions $N_n^m(\mu)$:

$$
N_2^1(\mu) = \sum_{l=0}^\infty \frac{l+\frac{1}{2}}{[\mu^2 + (l+\frac{1}{2})^2]^2}, \quad \text{and} \quad N_3^i(\mu) = \sum_{l=0}^\infty \frac{(l+\frac{1}{2})^i}{[\mu^2 + (l+\frac{1}{2})^2]^3}, \tag{B.3}
$$

where $i = 1, 3$. Using the formula (5.1.26.23) from Ref. [20] we find

$$
N_2^1(\mu) = -\frac{i}{4\mu}\left[\psi'\left(\frac{1}{2} - i\mu\right) - \psi'\left(\frac{1}{2} + i\mu\right)\right], \tag{B.4}
$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. It is also easily to see that

$$
N_3^1(\mu) = -\frac{1}{4\mu}\frac{dN_2^1(\mu)}{d\mu} \quad \text{and} \quad N_3^3(\mu) = N_2^1(\mu) - \mu^2 N_3^1(\mu).
$$

Using the asymptotical properties of the digamma function (see, for example, Ref. [21]) we can obtain for large values of μ the following asymptotics:

$$
\begin{aligned}
N_2^1(\mu) &= \frac{1}{2(mr)^2} + \mathrm{O}((mr)^{-4}), \qquad N_3^1(\mu) = \frac{1}{4(mr)^4} + \mathrm{O}((mr)^{-6}), \\
N_3^3(\mu) &= \frac{1}{4(mr)^2} + \mathrm{O}((mr)^{-4}).
\end{aligned} \tag{B.5}
$$

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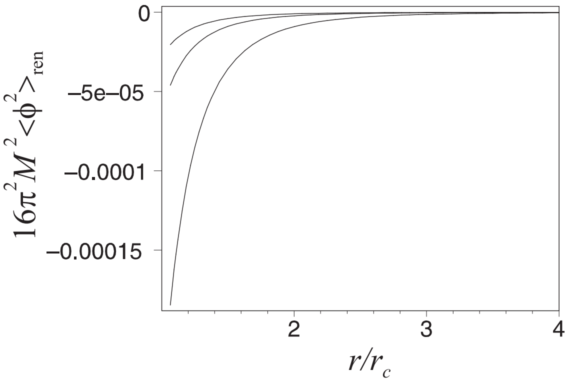

FIG. 1. The curves in this figure display the value of $\langle\phi^2\rangle$ for a massive scalar field in the Schwarzschild spacetime. From bottom to top the curves correspond to the values $\Lambda = r_{\rm g}/r_{\rm c} = 10, 20, 30$.

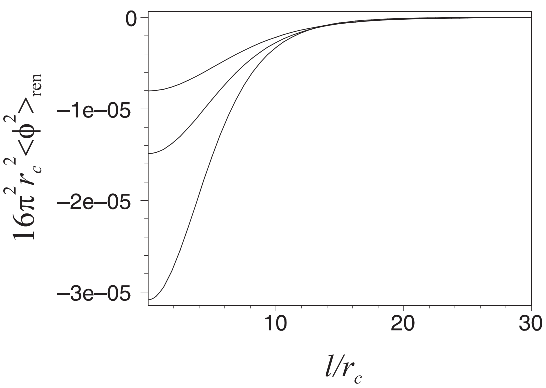

FIG. 2. The curves in this figure display the value of $\langle\phi^2\rangle$ for a massive scalar field in the wormhole spacetime. From bottom to top the curves correspond to the values $\rho_0 = r_0/r_{\rm c} = 10, 12, 14$.